Paying Attention to Proportional Reasoning

K–12

Support Document for Paying Attention to Mathematical Education

Contents

Paying Attention to Proportional Reasoning

> *"All ability to reason using proportional relationships is a complex process that develops over an extended period of time. It takes many varied physical experiences to develop an understanding of what a proportional relationship is and then more time to gain the ability to deal with it abstractly."*
>
> (Cordel, & Mason, 2000)

Paying Attention to Mathematics provided an overview of what it would take to help Ontario students make – and sustain – gains in their learning and understanding of mathematics. It outlined seven foundational principles for planning and implementing improvements and gave examples of what each principle would involve.

This document gets more concrete by focusing on a particular area of mathematics. Future support documents will explore other key topics in mathematics teaching and learning.

Proportionality permeates mathematics and is often considered the foundation to abstract mathematical understanding. We hope this document serves to spark learning about this complex and important topic, both with colleagues and with students in your schools and classrooms.

SEVEN FOUNDATIONAL PRINCIPLES FOR IMPROVEMENT IN MATHEMATICS, K–12

- Focus on mathematics.
- Coordinate and strengthen mathematics leadership.
- Build understanding of effective mathematics instruction.
- Support collaborative professional learning in mathematics.
- Design a responsive mathematics learning environment.
- Provide assessment and evaluation in mathematics that supports student learning.
- Facilitate access to mathematics learning resources.

What Is Proportional Reasoning?

Students use proportional reasoning in early math learning, for example, when they think of 8 as two fours or four twos rather than thinking of it as one more than seven. They use proportional reasoning later in learning when they think of how a speed of 50 km/h is the same as a speed of 25 km/30 min. Students continue to use proportional reasoning when they think about slopes of lines and rates of change.

The essence of proportional reasoning is the consideration of number in relative terms, rather than absolute terms. Students are using proportional reasoning when they decide that a group of 3 children growing to 9 children is a more significant change than a group of 100 children growing to 150, since the number tripled in the first case; but only grew by 50%, not even doubling, in the second case.

Although the Ontario curriculum documents for mathematics do not reference the term proportional relationships until Grade 4, activities in the primary grades support the development of *proportional reasoning*. For example, if we ask students to compare the worth of a group of four nickels to the worth of a group of four pennies, we are helping them to develop proportional reasoning. In the junior and intermediate grades, students work directly with fractional equivalence, ratio, rate and percent.

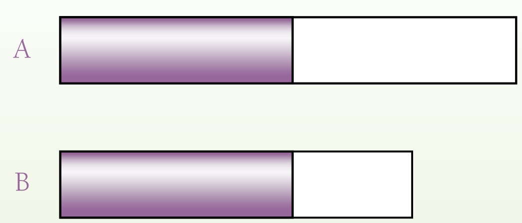
Much formal work on proportion is completed in Grades 9 or 10, but students in higher grades often compare proportional to non-proportional situations. They continue to use proportional reasoning when they work with trigonometry and with scale diagrams, as well as in other situations.

Proportional reasoning involves thinking about relationships and making comparisons of quantities or values. In the words of John Van de Walle, "Proportional reasoning is difficult to define. It is not something that you either can or cannot do but is developed over time through reasoning … It is the ability to think about and compare multiplicative relationships between quantities" (2006, p. 154).

Proportional reasoning is sometimes perceived as only being the study of ratios, rates and rational numbers such as fractions, decimals and percents, but it actually permeates all strands of mathematics. For example, proportionality is an important aspect of measurement, including unit conversions and understanding the multiplicative relationships of dimensions in area and volume.

Which shape is more purple?

An example of proportional reasoning in area measurement.



Adapted from Marian Small (2008, p. 254)

Giving students non-numerical representations which require qualitative reasoning can evoke rich discussions about proportionality. For examples: Small (2008), Van de Walle (2005) and *Continuum & Connections: Big Ideas and Questioning: Proportional Reasoning K–12*.

Why Is It Important?

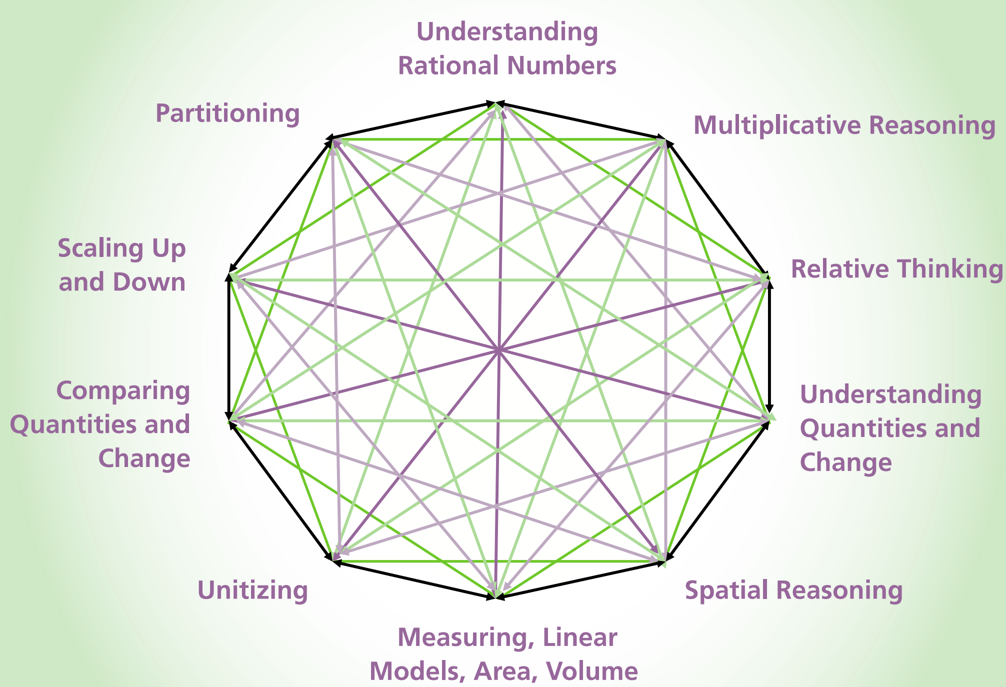
Beyond the mathematics classroom, proportional reasoning is evident in other subject areas like science, music and geography, as well as in everyday activities. People use proportional reasoning to calculate best buys, taxes and investments, to work with drawings and maps, to perform measurement or monetary currency conversions, to adjust recipes or to create various concentrations of mixtures and solutions.

The ability to think and reason proportionally is one essential factor in the development of an individual's ability to understand and apply mathematics. Susan Lamon estimates that over 90% of students who enter high school cannot reason well enough to learn mathematics and science with understanding and are unprepared for real applications in statistics, biology, geography or physics (Lamon, 2005, p. 10). While students may be able to solve a proportion problem with a memorized procedure, this does not mean that they can think proportionally.

Exploring Some Key Concepts

Proportional reasoning is a complex way of thinking and its development is more web-like in nature than linear. Students do not think through an identical concept in exactly the same way so there are myriad possibilities at play when developing the ability to reason proportionally (see the web below). It is important to give students of all ages a variety of proportional reasoning experiences and to encourage them to make conjectures, devise rules and generalize their learning.

Some Interconnected Proportional Reasoning Concepts



Experience and development in one area evokes change and growth in another.

Here are a few examples that provide further clarification of some proportional reasoning concepts and highlight some interconnections.

UNITIZING AND SPATIAL REASONING

These concepts involve being able to envision a particular quantity in same-sized groupings (or quantity sets) and being able to conceptualize them as units. For example, a dime can be considered as both 1 dime and 10 cents simultaneously. The unit is the dime, so 3 dimes represent 3 units – each worth 10 cents.

Other Examples of Unitizing and Spatial Reasoning

The rectangle can be considered as both 1 rectangle (unit) and 4 squares simultaneously. There are 5 units, each with 4 squares.

The nest can be considered as both 1 nest (unit) and 3 eggs simultaneously. There are 4 units, each with 3 eggs.

Why is this important?

It is critical to spend time developing unitizing since "the ability to use composite units is one of the most obvious differences between students who reason well with proportions and those who do not" (National Research Council, 2001, p. 243). Unitizing is the basis for multiplication and our place value system which requires us to see ten units as one ten and one hundred units as ten tens. As Cathy Fosnot emphasizes, this is complex since unitizing ten things as one thing almost negates children's original understanding of number (Fosnot & Dolk, 2001, p. 11). It is therefore not surprising that a robust understanding of place value has been found by some researchers to not fully mature until fifth grade (Brickwedde, 2011, p.13). Spatial visualization and reasoning are key components of unitizing. Spatial visualization allows the student to understand the unit as equal intervals of distance.

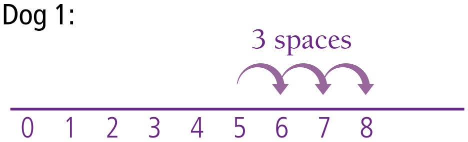
MULTIPLICATIVE THINKING

This concept involves reasoning about several ideas or quantities simultaneously. It requires thinking about situations in relative rather than absolute terms. Consider the following problem. *If one dog grows from 5 kilograms to 8 kilograms and another dog grows from 3 kg to 6 kg, which dog grew more?* When a student is thinking in absolute terms or additively, she/he might answer that both dogs grew by the same amount. When a student is thinking in relative terms, she/he might argue that the second dog grew more since he doubled his previous weight, unlike the first dog who would have needed to be 10 kg to grow by the same relative amount. While both answers are viable, it is the relative (multiplicative thinking) that is necessary for proportional reasoning.

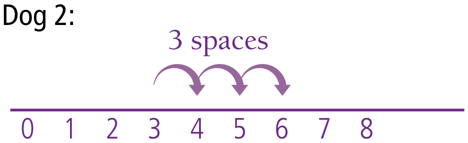
Here is student thinking that is absolute:

The first dog grew by 3kg. The second dog grew by 3 kg. They grew the same amount."

Dog 1:

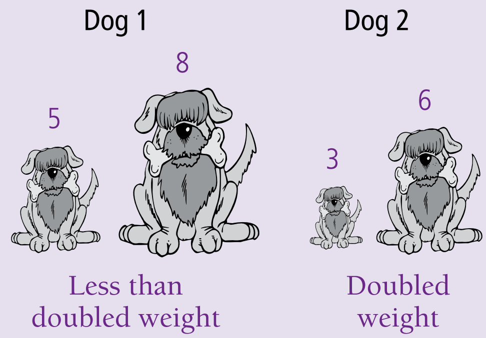


Dog 2:



Here is student thinking that is relative:

The second dog grew more since he doubled his previous weight, unlike the first dog who would have needed to be 10 kg to grow by the same relative amount.



Why is this important?

Helping students bridge from additive to multiplicative thinking is complex but starts early. It forms the backbone of the mathematics curriculum and includes important and interconnected ideas such as multiplication, division, fractions, decimals, ratios, percentages and linear functions. It requires time, a variety of situations and opportunities to construct their understanding in multiple ways.

UNDERSTANDING QUANTITY RELATIONSHIPS AND CHANGE

These concepts involve thinking about how quantities relate, co-vary or change together (how a variation in one quantity coincides with the variation in another). Consider the following example: Every time you buy one pack of gum, you get 5 sports cards. For 10 packs, you get 50 cards. The total number of cards you get (C) is dependent upon the number of packs of gum you purchase (P) so that $C = 5 \times P$ or $C = 5P$. Sometimes only part of the quantity relationship is proportional. For example, a cell phone plan has a fixed monthly base cost and a cost per minute of use. The base cost is a constant. The "minutes of use" is a variable that has proportionate cost based on use. Intermediate and Senior students study this as the linear relationship, $y=mx+b$.

Why is this important?

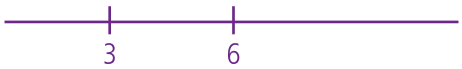
The concepts of quantity relationships and change play a central role in our daily lives and are essential for developing algebraic reasoning.

PARTITIONING, MEASURING, UNIT RATES AND SPATIAL REASONING

These concepts are linked to proportional reasoning because they involve reasoning about equal splitting of a whole, determining relative location and comparing measures of two different things through strategies such as guess and check, measuring, successive division of a unit and/or calculating differences. This can involve reasoning about two data points to find a third.

Consider the following examples:

Example 1:



Where does 7 go?

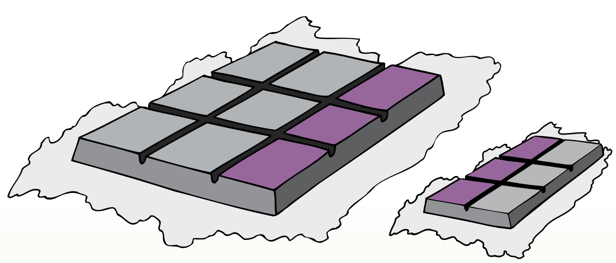
Example 2: Three tennis balls cost $3.75 and six tennis balls cost $7.50.
How much do seven tennis balls cost?

Why is this important?

Understanding equal portions, relative values and rates are cornerstones to mathematics and daily living. Partitioning activities that require the use of equivalence strategies provide educators with visual, verbal and symbolic evidence that children are bridging the gap between additive and multiplicative reasoning (Lamon, 1996, p.190). Partitioning and rate activities should therefore be amplified and extended as students progress through the grades, increasing the variety of contexts and complexity of situations.

UNDERSTANDING RATIONAL NUMBERS

Rational numbers are numbers that can be expressed as fractions; they can be challenging for students to grasp since they must see numbers expressed in relation to other numbers rather than as a fixed quantity, like whole numbers. Consider the following problem: Describe a situation when one third is greater than one half. This can be challenging if students have not had experience with comparing fractions in relation to their wholes. For example, one third of a jumbo-sized chocolate bar can be much larger than one half of a mini chocolate bar.

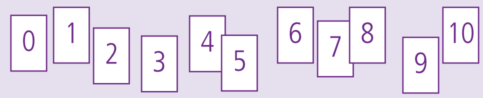
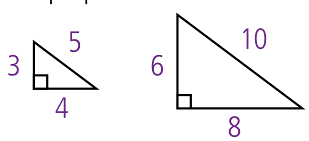
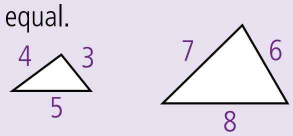


Why is this important?

Fractions underpin the understanding of algebra and the density of numbers (between any two numbers, there are infinitely more numbers). They are necessary in our lives as we read scaled maps, invest money and calculate discounts.

Is It or Isn't It Proportional?

Students have difficulty discriminating proportional from non-proportional situations and apply proportional strategies and procedures to problems that are additive, or vice versa. Exposing students to both types of problems and allowing them to discuss the differences will contribute to their further development of proportional reasoning.

<table>
<tr><th>Proportional Situations</th><th>Non-Proportional Situations</th></tr>
<tr><td>Determining equal intervals on a number line from 0 to 10.
</td><td>Ordering numbers from 0 to 10.
</td></tr>
<tr><td>Same unit rates for the price of objects: 6 pencils cost $2.40 and 12 pencils cost $4.80.</td><td>Different unit rates for the price of objects: 6 pencils cost $2.40 and 12 pencils cost $4.00.</td></tr>
<tr><td>Maps drawn to scale: Proportions are consistent.</td><td>Adding two units to the length and width of a rectangle.</td></tr>
<tr><td>The triangles shown below are similar since all side lengths are proportional and all angles equal.
</td><td>The triangles shown below are not similar since side lengths are not proportional and not all angles are equal.
</td></tr>
</table>

How Can We Get Started?

SOME TOP TIPS

- Provide students with proportional situations that span a wide range of contexts and relate to their world.
- Offer problems that are both qualitative and quantitative in nature. Qualitative problems (e.g., Which shape is more blue?) encourage students to engage in proportional reasoning without having to manipulate numbers.
- Help students distinguish between proportional and non-proportional situations.
- Encourage discussion and experimentation in predicting and comparing ratios.
- Help students relate proportional reasoning to what they already know. For example, connect how unit fractions and unit rates are very similar.
- Recognize that mechanical procedures for solving proportions do not develop proportional reasoning and that students need to be flexible in their thinking and acquire many strategies.

(Adapted from Van de Walle, 2001, p. 262)

PROPORTIONAL REASONING ACROSS STRANDS AND GRADES

Here is a sample of mathematics tasks that involve proportional reasoning drawn in part from *Continuum & Connections: Big Ideas and Questioning: Proportional Reasoning K–12*.

<table>
<tr><th>Strand</th><th>Primary/Junior/Intermediate</th><th>Intermediate/Senior</th></tr>
<tr><td>Number Sense and Numeration</td><td>How many different ways can you cut a sandwich in half?

How do you know that if you skip counted to find the number of socks worn in the class, it probably would not be 51?

Eric says that 8/8 is greater than 4/4 because there are more pieces.Sylvia says 4/4 is greater because the pieces are bigger. What do you think?

A store is having a sale. Is it more helpful to know that you get $10 off or 10% off? Explain.</td><td rowspan="3">Give an example of a linear relationship that is not proportional.

How does $y = 4f(x)$ compare to $y = f(x)$?

When using simple interest, what effect does doubling the rate have on your interest earned? Compound interest?

Why might it be useful to report fuel efficiency as L/100 km? Would it be just as useful to report it as km/L?

How does doubling the diagonal of a square affect its perimeter and area?

One cylinder has double the volume of another. How could the radii and heights be related?

Given vector u is (a,b) and vector v is (3a, 3b). Is the magnitude of vector v three times as large as vector u? Justify your reasoning.

Sarah claims that when two triangles have one angle the same size, the triangles have proportional sides. Do you agree? Why or why not?</td></tr>
<tr><td>Measurement</td><td>Construct a tool for measuring heights and lengths.

If the table is 6 orange Cuisenaire rods long, how long would it be measured in yellow rods?

A rectangle and a parallelogram share a base. The parallelogram is twice as tall. If you know the area of the rectangle, can you figure out the area of the parallelogram?

One circle has double the radius of another. What is true about the relationship of their perimeters? areas?</td></tr>
<tr><td>Geometry and Spatial Sense</td><td>Using pattern blocks, how many green triangles do you need to cover one yellow hexagon? Two hexagons? Five hexagons?

What are the greatest/least number of pattern block shapes needed to compose a larger shape?

How can you create a similar rectangle by reducing or enlarging its size?

How can you use graph paper to enlarge a picture?</td></tr>
</table>

Continued on next page.

Strand	Primary/Junior/Intermediate	Intermediate/Senior
Patterning and Algebra	Create a growing or shrinking pattern using pennies or nickels starting with 20 cents. Create a number pattern involving multiplication given a rule expressed in words. Describe pattern rules that generate patterns by multiplying or dividing by a constant to get the next term. Model real-life relationships involving constant rates. Represent linear patterns graphically using a variety of tools.	The method of "cross multiplication" is often used to solve a proportion problem. The method is illustrated $\frac{x}{4} \times \frac{6}{5}$ then we cross multiply and obtain the equation $5x = 24$ which we solve and obtain $x = \frac{24}{5}$ (or 4.8) Provide a good argument to show that cross multiplication is a valid method for solving a proportion problem.
Data Management and Probability	The ☺ represents 5 people on the pictograph. How many people are represented by 4 ☺ ? If you have a one in three chance of winning, how many times would you likely win in 24 trials? Explain how different scales used on graphs can influence the conclusions drawn from the data. Research and report on probability situations expressed in fraction, decimal, and percent form (batting averages, weather forecasts).	You are creating a graph of a function. The minimum value for y is 0 and the maximum for the domain of interest is 422. What scale should you use on the y-axis? Why that scale? A poll is accurate to within 3 percentage points 19 times out of 20. What do these numbers tell you?

MINISTRY RESOURCES

Continuum & Connections: Big Ideas and Proportional Reasoning K–12.

Identifies the big ideas of proportional reasoning from K–12, maps curricular connections that address proportional reasoning across the grades, provides examples of open questions, parallel tasks and three-part lesson plans. Identifies related resources.

http://www.edugains.ca/resources/LearningMaterials/ContinuumConnection/BigIdeasQuestioning_ProportionalReasoning.pdf

Guides to Effective Instruction in Mathematics, Kindergarten to Grade 6: Number Sense and Numeration.

http://eworkshop.on.ca/edu/core.cfm

TIPS4RM Grades 7–12

Three-part lesson plans and supports for Grade 7 through Grade 12.
http://www.edugains.ca/newsite/math2/tips4rm.html

Gap Closing materials.

Intervention materials designed for students who need additional support in mathematics. Accompanied by facilitator guides.

- *Gap Closing Junior/Intermediate*
  Module 1 focuses on representing fractions and Module 2 focuses on comparing fractions.
  http://www.edugains.ca/newsite/math2/gapclosing.html
- *Gap Closing Intermediate/Senior*
  Module 1 focuses on comparing, adding, subtracting, multiplying and dividing fractions. This module also focuses on relating situations to fraction operations.
  http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/1-Fractions_FG_IS.pdf
- ePractice (www.epractice.ca). Provides students with additional interactive practice activities.

Critical Learning Instructional Paths Supports (CLIPS)

Interactive activities with immediate feedback.

- Fractions – Exploring Part-Whole Relationships. www.mathclips.ca

Being Responsive to Student Thinking

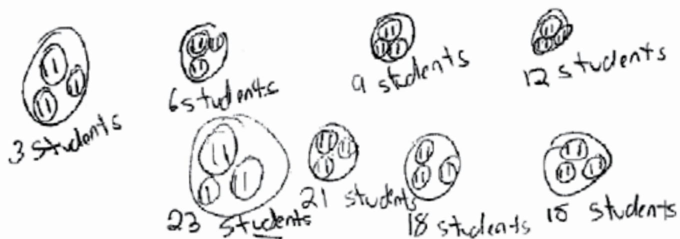
Some annotated student solutions to problems requiring proportional reasoning are reproduced below to help reveal student thinking – both misconceptions and the various strategies they use. Understanding and being responsive to student thinking can help to plan next steps.

GRADE 3 EQAO, 2008–09, QUESTION 28, CODE 30

Annotation: Student demonstrates a considerable understanding of the relationships between the important elements of the problem; correctly identifies number of slices required (46) for each student to get 2 slices and groups slices into pizzas but does not deal with the extra pizza needed to get 46 slices.

A Grade 3 class wins a pizza party for reading the most books in September. There are 23 students in the class and each student will get 2 slices of pizza. If each pizza has 6 slices, how many pizzas should the class buy?

Show your work.



The class should buy ___7___ pizzas.

Assets: Student is demonstrating a solid understanding of unitizing by creating groups of two (slices of pizza) as representing the amount of pizza each person gets. The student recognizes a 2 pieces of pizza: 1 person relationship.

The student sets these units of 2 into groups of 3 to represent all 6 slices in a given pizza.

Finally, the student attempts to account for the pizza for all students by generating multiple diagrams of the 6 pizza slices. The student counts by 3's beginning on the top left: 3 students, 6 students, 9 students, 12 students, 15, 18, 21, and then writes 23.

Wondering: Did the student use the diagram as his/her problem-solving strategy or is the diagram simply a way to communicate his/her thinking after the fact?

Observation: This student is holding multiple ideas in his/her head simultaneously or is tracking these layers of information strategically. He/she is considering all of the following:

- the number of slices of pizza per person
- the number of people
- the number of slices per pizza
- the total number of pizzas

Challenge: The student did not account for the 8th pizza required.

Access EQAO Grade 3 Question 28 at
http://www.eqao.com/pdf_e/09/3e_Math_WebRelease_ScoringGuide.pdf

GRADE 6 EQAO, 2010, QUESTION 28, CODE 30

EQAO Annotation: Student demonstrates considerable understanding of the relationships between important elements of the problem; calculates and compares the cost for 180 minutes at each company but multiplies by 3 instead of 4 for Company B; conclusion matches calculations.

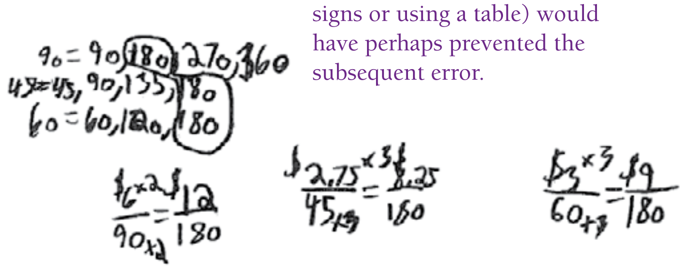
The rates for Internet use offered by three companies are shown below.

- Company A: $6.00 for every 90 minutes of use
- Company B: $2.75 for every 45 minutes of use
- Company C: $3.00 for every 60 minutes of use

Which company offers the lowest rate per minute?

Show your work.

Aligning these lines vertically (either by lining up the equal signs or using a table) would have perhaps prevented the subsequent error.



Company B offers the lowest rate per minute.

Assets: The student has selected an appropriate strategy. She/he established a common unit of 180 minutes for comparison and then calculated the cost for that common unit.

It was easier to follow the student's thinking and identify the error made because of the effective communication shown by including the multipliers in the ratios. As well, the inclusion of the dollar signs increased clarity.

Wondering: Did the student start with minutes rather than cost because she/he recognized that the minutes were "nicer numbers" or because the question focused on minutes?

This solution is not one that we would have expected. Even though the final answer is incorrect, the solution clearly shows correct proportional reasoning.

It is not obvious whether the student reflected on the reasonableness of the answer. A quick comparison of Company B to Company A would have supported this conclusion but an equally quick comparison of Company B to Company C may have raised questions about the correctness of the solution.

Observation: We noticed that the numbers selected for the question would easily allow students to check the reasonableness of their answers. We also noticed that this question could be solved using a variety of strategies.

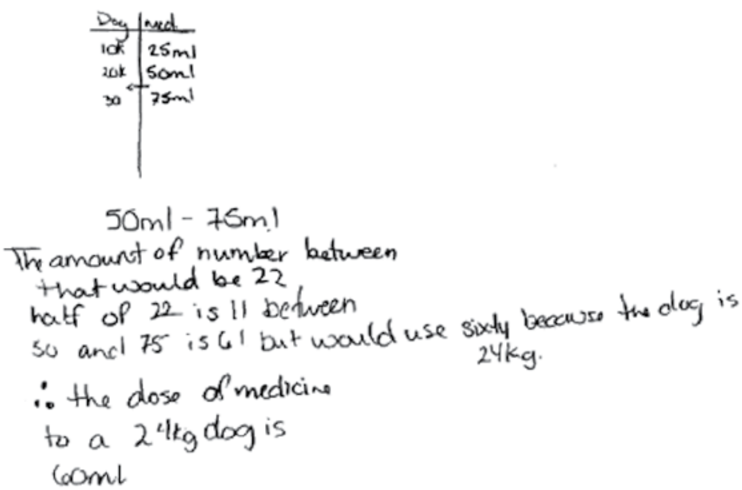
Challenge: Circling the 180s clarified the purpose of the first three lines. However, use of words to introduce the strategy would have helped.

Access Grade 6 Question 28 at:
http://www.eqao.com/pdf_e/10/6e_Math_WebRelease_ScoringGuide.pdf

GRADE 9 APPLIED EQAO SPRING 2008 QUESTION 6

Clarence's Quandary

Clarence works at a veterinarian's office. He needs to give a dose of medicine to a 24 kg dog. The recommended dosage for a dog that weighs 10 kg is 25 mL. Determine the dose Clarence should give to the 24 kg dog if the rate remains the same. Show your work.



Assets: The student successfully sets up a table to compare dog weight to dosage and correctly lists three equivalent ratios (as the dog's weight doubled from 10 to 20 kg, the student doubled dosage, and as the weight tripled to 30 kg, the student also tripled the dosage), indicating proportional reasoning (multiplicative thinking).

The student attempts to work a half ratio between 20 and 30 to find the dosage for a 25 kg dog by finding halfway point between 50 ml and 75 ml.

She/he adequately explains the halving strategy but does not clearly explain that she/he is finding the dosage for a 25 kg dog.

Wondering: When filling out the table, was the student using additive thinking, filling it in vertically by following the two separate patterns? 10, 20, 30 … and 25, 50, 75 …, or seeing the relationship between the horizontal number pairs?

How did the student calculate 22?

What would the student have done if she/he used the correct difference of 25 (between50 and 75) since it is odd?

Observation: While the student incorrectly calculates distance between 50 and 75, using 22 rather than 25, she/he successfully applies the strategy of using his/her halfway point of 11 (between 22) to calculate the dosage for a 25 kg dog. The strategy reveals an understanding of proportional reasoning and multiplicative thinking involving halves.

Challenges: The student uses additive reasoning to calculate the weight for a 24 kg dog, reasoning that a dog that is 1 kg less than 25 kg would need 1 ml less of medication. This could indicate that the proportional reasoning is still fragile when numbers are less friendly and do not conform to halving, doubling or tripling.

Access Clarence's Quandary at: http://www.eqao.com/pdf_e/08/9e_App_0608_Web.pdf

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Printed on recycled paper

ISBN 978-1-4606-0213-3 Print

ISBN 978-1-4606-0214-0 PDF